Introduction to Mathematics and Modeling

lecture 7

Points, lines and planes

UNIVERSITY OF TWENTE.

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1. Lines in $\mathbb{R}^2$
2. Section 12.5: lines and planes in space
3. Application: perspective projection
Convention

- From now on we will identify points with terminal points of vectors in standard position:
  \[ P = \mathbf{v} \]

- We will abandon the notation \( \langle x_1, \ldots, x_n \rangle \) and use \((x_1, \ldots, x_n)\) instead.
### Definition

A line in \( \mathbb{R}^2 \) is defined by an equation of the form

\[
\ell : ax + by = c
\]  

(*)

with \( a, b \) and \( c \) real numbers.

- The line \( \ell \) consists of the points that satisfy equation (*):

\[
\ell = \{(x, y) \mid ax + by = c\}.
\]

- The line \( \ell \) is the **solution set** of equation (*).
Parametrisation

**Definition**

A *parametrisation* of the line $\ell$ is a function $r : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $r(t)$ reaches all points of $\ell$ while $t$ runs through all real numbers.

- The number $t$ is called the **parameter**.
- The line $\ell$ is the set of all points $r(t)$:
  \[ \ell = \{ r(t) \mid t \in \mathbb{R} \} \]
- The function $r(t)$ has two components that both depend on $t$:
  \[ r(t) = (x(t), y(t)) \]
- Functions like $r$ with values in $\mathbb{R}^n$ are called **vector functions**.
Example

Given is the line $\ell: 2x + 3y = 6$. Find a parametrisation of $\ell$. 
Example

Find an equation for the line

\[ \ell : (3t, 2 - 2t), \quad t \in \mathbb{R}. \]
**Theorem**

For every line $\ell$ there exist numbers $p_1$, $p_2$, $v_1$ and $v_2$ such that

$$r(t) = (p_1 + v_1 t, p_2 + v_2 t) \quad t \in \mathbb{R}.$$  

- Write $r(t)$ as follows:
  $$r(t) = (p_1, p_2) + t(v_1, v_2).$$
- The vector $p = (p_1, p_2)$ is called a **support vector** of $\ell$.
- The vector $v = (v_1, v_2)$ is called a **direction vector** of $\ell$.
- Define $q = r(1)$, then
  $$r(1) = p + v, \quad \text{dus} \quad v = q - p.$$  
- The **parametrised vector form** of $\ell$ is
  $$\ell: r(t) = p + tv \quad t \in \mathbb{R}.$$
Example

Find a support- and a direction vector of the line $\ell: 2x + 3y = 6$, and find a parametrised vector form of $\ell$.
Definition

Let \( \mathbf{p} \) and \( \mathbf{v} \neq \mathbf{0} \) be vectors. The parametrised vector form of the line through \( \mathbf{p} \) and parallel to \( \mathbf{v} \) is

\[
\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbb{R}.
\]

- The vector \( \mathbf{p} \) is called a support vector and the vector \( \mathbf{v} \) is called a direction vector of the line.
- If \( \mathbf{r}(t) = (f(t), g(t), h(t)) \), then the equations

\[
\begin{align*}
  x &= f(t), \\
  y &= g(t), \\
  z &= h(t)
\end{align*}
\]

are called the parametric equations of the line.
Example

Find the parametric equations of the line \( \ell \) through \((-2, 0, 4)\) in the direction
\[
v = 2i + 4j - 2k
\]
\[
= (2, 4, -2).
\]
Example

Find the parametric equations of the line \( \ell \) through \( P = (-3, 2, -3) \) and \( Q = (1, -1, 4) \).
Summary

- A parametrisation of the line through a point $P$ parallel to a vector $v \neq 0$ is
  \[ p + tv, \quad t \in \mathbb{R}, \]
  with support vector $p = \overrightarrow{OP}$ and direction vector $v$.

- A parametrisation of the line through two points $P$ and $Q$ is
  \[ p + tv, \quad t \in \mathbb{R} \]
  with support vector $p = \overrightarrow{OP}$ and direction vector $v = \overrightarrow{PQ}$.

Warning

Parametrisations are not unique:

- Every point on the line can be chosen as support vector.
- Every non-zero vector parallel to the line can be chosen as direction vector.
Suppose two lines $\ell$ and $m$ have parametrised vector forms $p + tv$ and $q + sw$ respectively.

An intersection is found if there are values for $t$ and $s$ such that

$$p + tv = q + sw. \quad (*)$$

Since vector equations in $\mathbb{R}^3$ yield three equations, equation $(*)$ may fail to have a solution, even if $\ell$ and $m$ are not parallel.

Non-parallel lines that do not intersect are called skew.
Example

Let \( \ell \) be the line with support vector \((-3, -3, 1)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.
Example

Let \( \ell \) be the line with support vector \((-3, -3, 0)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.
### Definition

A plane in \( \mathbb{R}^3 \) is defined by an equation of the form

\[
M : ax + by + cz = d
\]

with \( a, b, c \) and \( d \) real numbers.

### Examples:

- The plane \( M_1 \) defined by
  \[
  M_1 : x + y + z = 1
  \]
  passes through the points \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\).

- The plane \( M_2 \) defined by
  \[
  M_2 : x + y + z = 0
  \]
  passes through \( O \) and is parallel to \( M_1 \).

- The plane \( M_3 \) defined by
  \[
  M_3 : 2y = 3
  \]
  is the plane through \((0, 3/2, 0)\) parallel to the \(xz\)-plane.
Definition

A support vector of a plane $M$ is a vector $\mathbf{p} = \vec{OP}$ with $P$ a point of $M$.

Suppose $M$ is defined by $ax + by + cz = d$, and let $P = (x_0, y_0, z_0)$ be a point in $M$, then $ax_0 + by_0 + cz_0 = d$, hence

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

for all $(x, y, z,)$ in $M$.

Definition

The equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is called the vector equation of $M$. 
Definition

A normal vector of a plane $M$ is a vector $n \neq 0$ that is perpendicular to $M$.

- Let $M$ be a plane defined by the vector equation 
  \[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \]
  then for all $(x, y, z,)$ in $M$:
  \[ (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0, \]
  \[ (a, b, c) \cdot ((x, y, z) - (x_0, y_0, z_0)) = 0. \]

- Define $x = (x, y, z)$, $p = (x_0, y_0, z_0)$ and $n = (a, b, c)$, then
  \[ n \cdot (x - p) = 0 \quad \text{for all } x \text{ in } M. \]

Definition

The equation $n \cdot (x - p) = 0$ is called the normal equation of $M$. 
Theorem

Let $M$ be defined by the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, where $\mathbf{n}$ is a normal vector of $M$, and let $\mathbf{p} = (x_0, y_0, z_0)$ be a support vector. If $X = (x, y, z)$ is a point of $M$ then $\mathbf{n} \perp \overrightarrow{PX}$.

- Note that $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$. 

![Diagram](image)
Example

Find an equation of the plane \( M \) through \((-3, 0, 7)\) orthogonal to \( \mathbf{n} = (5, 2, -1) \).
Example

Find a normal equation for the plane $M : y - 2z = 4$. 

Unlike lines in $\mathbb{R}^2$, lines in $\mathbb{R}^3$ cannot be described by one equation: a linear equation $ax + by + cz = d$ describes a **plane**.

In order to describe a line you need *two* equations:

$$\begin{cases} ax + by + cz = d \\ px + qy + rz = s \end{cases}$$

Regard a line as the intersection of two planes:
Example

Give a parametrisation of the line described by the equations

\[
\begin{align*}
x + y - 2z &= -1 \\
2x - y + z &= 2
\end{align*}
\]
Check your answer!
Example

The line $\ell$ is defined by the parametrisation

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t, \quad t \in \mathbb{R}.$$  

Find the intersection of $\ell$ and the plane $3x + 2y + 6z = 6$. 

Parametrise the line $\ell$ as follows:

\[ \ell : \mathbf{r}(t) = (x_0, 0, 0) + t(x_1 - x_0, y_1, z_1), \quad t \in \mathbb{R}. \]

The intersection of $\ell$ and the $yz$-plane is $P = \mathbf{r}(t_0)$ with $t_0 = \frac{x_0}{x_0 - x_1}$.

For $P = (0, y, z)$ we have

\[ y = t_0 y_1 = \frac{x_0 y_1}{x_0 - x_1} \quad \text{and} \quad z = t_0 z_1 = \frac{x_0 z_1}{x_0 - x_1}. \]